

Construction of Affine and Conformal Affine Toda Solitons by Hirota's method ¹

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ABSTRACT

In this talk we report some results about the construction of soliton solutions for the Affine and Conformal Affine Toda models using the Hirota's method. We obtain new classes of solitons connected to the degeneracies of the Cartan matrix eigenvalues as well as to some particular features of the recursive scheme developed here. We obtain an universal mass formula for all those solitons. The examples of $SU(6)$ and $Sp(3)$ are discussed in some detail.

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1 Introduction

One of the main motivations for searching for solitons solutions in the Toda models comes from the interest in generalizing the well known soliton solutions in the celebrated sine-Gordon model which equation of motion in two dimensions reads:

$$\partial_- \partial_+ \varphi = \frac{q}{\bar{q}} \sin(2\bar{q}\varphi) \quad (1.1)$$

In this case the vacuum configuration is degenerate and this is responsible for the topological nature of the soliton solutions.

From the point of view of Toda theories the sine-Gordon model can be considered as the simplest example of the Affine Toda (AT) model whose equations of motion are given by:

$$\partial_- \partial_+ \varphi^a = \frac{1}{\bar{q}} \left(q^a e^{\bar{q} K_{ab} \varphi^b} - q^0 l_a^\psi e^{-\bar{q} K_{\psi b} \varphi^b} \right) \quad (1.2)$$

where $K_{ab} = 2\alpha_a \cdot \alpha_b / \alpha_b^2$ is the Cartan Matrix of \mathcal{G} , $a, b = 1, \dots, \text{rank } \mathcal{G} \equiv r$, ψ is the highest root of \mathcal{G} , $K_{\psi b} = 2\psi \cdot \alpha_b / \alpha_b^2$, l_a^ψ are positive integers appearing in the expansion $\frac{\psi}{\psi^2} = l_a^\psi \frac{\alpha_a}{\alpha_a^2}$, where α_a are the simple roots of \mathcal{G} and q^a , q^0 and \bar{q} are coupling constants. Indeed, the sine-Gordon model equation of motion is obtained if one considers \mathcal{G} as $su(2)$. In this case $K_{ab} = 2$ and φ has only one component. The algebraic structure of the Affine Toda models is given by a loop algebra associated to \mathcal{G} .

Another hierarchy of Toda theories is obtained from the AT ones by adding two extra fields η and ν so that [1, 2]:

$$\partial_- \partial_+ \varphi^a = \frac{1}{\bar{q}} \left(q^a e^{\bar{q} K_{ab} \varphi^b} - q^0 l_a^\psi e^{-\bar{q} K_{\psi b} \varphi^b} \right) e^{\bar{q}\eta} \quad (1.3)$$

$$\partial_- \partial_+ \eta = 0 \quad (1.4)$$

$$\partial_- \partial_+ \nu = \frac{2}{\psi^2} \frac{q^0}{\bar{q}} e^{-\bar{q} K_{\psi b} \varphi^b} e^{\bar{q}\eta} \quad (1.5)$$

are the equations of motion of the so-called Conformal Affine Toda model (CAT), related to Kac-Moody algebras. These equations are invariant under conformal transformations:

$$x_+ \rightarrow \tilde{x}_+ = f(x_+) \quad , \quad x_- \rightarrow \tilde{x}_- = g(x_-) \quad (1.6)$$

and

$$e^{-\varphi^a(x_+, x_-)} \rightarrow e^{-\tilde{\varphi}^a(\tilde{x}_+, \tilde{x}_-)} = e^{-\varphi^a(x_+, x_-)} \quad (1.7)$$

$$e^{-\nu(x_+, x_-)} \rightarrow e^{-\tilde{\nu}(\tilde{x}_+, \tilde{x}_-)} = \left(\frac{df}{dx_+} \right)^B \left(\frac{dg}{dx_-} \right)^B e^{-\nu(x_+, x_-)} \quad (1.8)$$

$$e^{-\eta(x_+, x_-)} \rightarrow e^{-\tilde{\eta}(\tilde{x}_+, \tilde{x}_-)} = \left(\frac{df}{dx_+} \right)^{\frac{1}{\bar{q}}} \left(\frac{dg}{dx_-} \right)^{\frac{1}{\bar{q}}} e^{-\eta(x_+, x_-)} \quad (1.9)$$

where f and g are analytic functions and B is an arbitrary constant. Therefore e^{φ^a} are scalars under conformal transformations and e^ν can also be arranged to be a scalar by setting $B = 0$ [3]. The AT models, on the other hand, are not invariant under conformal transformations.

From (1.3) it is noticed that the AT model can be understood as a CAT model when the conformal symmetry is in some sense gauge fixed [3]. Making a suitable choice of transformations :

$$f'(x_+) = e^{\eta_+(x_+)} , \quad g'(x_-) = e^{\eta_-(x_-)} \quad (1.10)$$

where $\eta_{\pm}(x_{\pm})$ are solutions of the η field, i.e., $\eta(x_+, x_-) = \eta_+(x_+) + \eta_-(x_-)$, the CAT model equations can be written as:

$$\tilde{\partial}_- \tilde{\partial}_+ \tilde{\varphi}^a = q^a e^{K_{ab} \tilde{\varphi}^b} - l_a^\psi q^0 e^{-K_{\psi b} \tilde{\varphi}^b} \quad (1.11)$$

$$\tilde{\partial}_- \tilde{\partial}_+ \tilde{\nu} = \frac{2}{\psi^2} q^0 e^{-K_{\psi b} \tilde{\varphi}^b} \quad (1.12)$$

and the new space time coordinates are determined in terms of the old ones through the given solution of η , i.e. $\tilde{x}_+ = \int^{x_+} dy_+ e^{\eta_+(y_+)}$, $\tilde{x}_- = \int^{x_-} dy_- e^{\eta_-(y_-)}$. Observe that in these new coordinates equations (1.11) are the AT equations of motion.

The equations of motion of both hierarchies can be written as a zero curvature condition:

$$\partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0 \quad (1.13)$$

In the case of AT models the gauge potentials A_{\pm} lie on a loop algebra associated to \mathcal{G} , whilst for the CAT models they lie on a Affine Kac-Moody algebra $\hat{\mathcal{G}}$.

The solitonic character of the AT model can be observed from the potential of the theory. Introducing, for convenience, the following vector:

$$\varphi \equiv \sum_{a=1}^r \frac{2\alpha_a}{\alpha_a^2} \varphi^a \quad (1.14)$$

the potential can then be written as

$$U(\varphi, \eta) = \sum_{j=0}^r \frac{4q^j}{\alpha_j^2} e^{\bar{q}(\alpha_j \cdot \varphi + \eta)} \quad (1.15)$$

where $\alpha_0 = -\psi$ is denoted as the extra simple root of the affine Kac-Moody algebra $\hat{\mathcal{G}}$.

This potential is invariant under the transformation

$$\varphi \rightarrow \varphi + \frac{2\pi i}{\bar{q}} \mu^v ; \quad \eta \rightarrow \eta + \frac{2\pi i}{\bar{q}} n \quad (1.16)$$

where μ^v is a coweight of \mathcal{G} , i.e. $\mu^v = \sum_{a=1}^r m_a \frac{2\lambda_a}{\alpha_a^2}$ where m_a and n are integers, and λ_a are the fundamental weights of \mathcal{G} . For \bar{q} purely imaginary the vacuum is infinitely degenerate, corresponding to the coweight lattice of \mathcal{G} . This generalizes to any simple Lie algebra the degenerate vacua of the sine Gordon model where the minima of the potential are identified with co-weight lattice of $SU(2)$. The degeneracy shown above indicates the existence of topological solitons in the AT and CAT models.

2 Mass and Charge of the solitons

The energy-momentum tensor of the CAT model can be made traceless due to scale invariance:

$$\Theta_{\rho\sigma}^{\text{CAT}} = \Theta_{\rho\sigma}^{\text{CAN}} - \bar{q} \left(\partial_\rho \partial_\sigma - g_{\rho\sigma} \partial^2 \right) \left(\sum_{a=1}^r \frac{2}{\alpha_a^2} \varphi^a + h\nu \right) \quad (2.1)$$

where Θ^{CAN} is the canonical energy-momentum tensor, and h is the Coxeter number of \mathcal{G} . It is easy to verify that $\Theta_\rho^{\rho\text{CAT}} = 0$ and $\Theta_\rho^{\rho\text{CAN}} = 2U(\varphi, \eta)$.

Consider a soliton type solution which can be put at rest at some Lorentz frame. Then the energy of the solution should be the mass of the soliton. Such mass should be proportional to some mass scale of the theory. Due to conformal (scale) invariance such mass scale does not exist in the CAT model and therefore the solitons are massless. For the AT model, however, such argument does not apply and the solitons are massive. In this case:

$$\Theta_{\rho\sigma}^{\text{AT}} = \Theta_{\rho\sigma}^{\text{CAN}} \big|_{\eta=0} \quad (2.2)$$

One then observes from (2.1) and (2.2) that the contribution to the soliton mass in the AT models comes from a total divergence term, which involves the CAT field ν . Indeed, denoting by M the soliton mass and v its velocity (in units of light velocity) one gets:

$$\begin{aligned} \frac{Mv}{\sqrt{1-v^2}} &= \int_{-\infty}^{\infty} dx \Theta_{01}^{\text{AT}} \\ &= \bar{q} \int_{-\infty}^{\infty} dx \partial_x \partial_t \left(\sum_{a=1}^r \frac{2}{\alpha_a^2} \varphi^a + h\nu \right) \\ &= \bar{q} \partial_t \left(\sum_{a=1}^r \frac{2}{\alpha_a^2} \varphi^a + h\nu \right) \Big|_{-\infty}^{\infty} \end{aligned} \quad (2.3)$$

Observe that this is a universal formula which can be used to determine the masses of the solitons [4, 5].

The topological charges of the solitons can still be introduced as :

$$Q \equiv \frac{\tilde{q}}{2\pi} \int_{-\infty}^{\infty} dx \partial_x \varphi = \frac{\tilde{q}}{2\pi} (\varphi(\infty) - \varphi(-\infty)) \quad (2.4)$$

where $i\tilde{q} \equiv \bar{q}$ and it will be a vector in the root space.

3 Construction of solitons: Hirota's Method

In order to follow Hirota's procedure [6] of constructing soliton solutions for nonlinear systems we introduce the τ functions as [7, 3, 8, 5]:

$$\varphi^a = \frac{1}{\bar{q}} \left(-\ln \frac{\tau_a}{\tau_0^{\psi_a}} + \vartheta_a \right) \quad \nu = \frac{1}{\bar{q}} \frac{2}{\psi^2} (\sigma - \ln \tau_0) \quad (3.1)$$

with $a = 1, 2, \dots, \text{rank-}\mathcal{G}$ and ϑ_a are constants depending on the coupling constants q^0 , q^a and \bar{q} (see [5] for more details).

For $\eta = 0$ the CAT equations (1.3) and (1.5) can be decoupled into

$$\Delta(\tau_j) = \beta l_j^\psi \left(1 - \prod_{k=0}^{\text{rank}\mathcal{G}} \tau_k^{-K_{jk}} \right) \quad (3.2)$$

$$\partial_+ \partial_- \sigma = \beta \quad (3.3)$$

with

$$\Delta(F) \equiv \partial_+ \partial_- \ln F = \frac{\partial_+ \partial_- F}{F} - \frac{\partial_+ F \partial_- F}{F^2} \quad (3.4)$$

and

$$\beta = \frac{q^j}{l_j^\psi} e^{K_{jk} \vartheta_k} \quad \text{for any } j = 0, 1, \dots, r \quad (3.5)$$

is a constant independent of the index j [3]. Now K_{jk} is the extended Cartan matrix of $\hat{\mathcal{G}}$ and in the calculation it was used the fact that l_j^ψ , $j = 0, 1, 2, \dots, \text{rank}\mathcal{G}$, with $l_0^\psi = 1$, constitute a null vector of the extended Cartan matrix, i.e., $\sum_{j=0}^r K_{ij} l_j^\psi = 0$.

The solution to (3.3) is given by

$$\sigma(x_+, x_-) = \beta x_+ x_- + F(x_+) + G(x_-) \quad (3.6)$$

with F and G being arbitrary functions. The solution for the τ 's, in Hirota's method spirit, is constructed using the following expansion:

$$\tau_i = 1 + \epsilon \tau_i^{(1)} + \dots + \epsilon^{N_i} \tau_i^{(N_i)} \quad (3.7)$$

with ansatz [7]:

$$\tau_i^{(n)} = \delta_i^{(n)} e^{n\Gamma} \quad (3.8)$$

$$\Gamma = \gamma(x - vt) + \xi \quad (3.9)$$

where $\delta_i^{(n)}$ are constant vectors to be found from Hirota's equations, γ and ξ are parameters of the solution. One can show that highest orders of τ 's constitute a null vector of the extended Cartan matrix [5]

$$K_{ij} N_j = 0 \quad (3.10)$$

Therefore $N_i = \kappa l_i^\psi$, where κ is some positive integer.

Substituting expansion (3.7) in (1.3) one finds in first order in ϵ :

$$L_{ij} \delta_j^{(1)} = \lambda \delta_i^{(1)} \quad (3.11)$$

where $L_{ij} = l_i^\psi K_{ij}$, and $\lambda = \frac{\gamma^2(1-v^2)}{4\beta}$. Therefore the parameters of the solution are restricted by the possible eigenvalues of the matrix L_{ij} .

The higher δ 's are determined from $\delta^{(1)}$ recursively through [5] :

$$\delta_i^{(n)} = S_{ij}^{(n)-1} V_j^{(n-1)} \quad (3.12)$$

with

$$S^{(n)} \equiv L - n^2 \lambda \quad (3.13)$$

and $V_j^{(n-1)} \equiv$ function of δ 's of order $(n-1)$ or less.

Higher δ 's are uniquely determined by $\delta^{(1)}$ except for the cases where L has two eigenvalues λ and λ' satisfying

$$\lambda = n^2 \lambda' \quad \text{for some integer } n \quad (3.14)$$

This kind of degeneracy appears only for $SU(6p)$ and $Sp(3p)$, with p a positive integer.

Once the δ 's are determined one is able to write φ explicitly and so the soliton masses are calculated using the universal formula (2.3):

$$M = \frac{4h\kappa}{\psi^2} m \sqrt{\lambda} \quad (3.15)$$

where $m \equiv \sqrt{\beta}$, h is the Coxeter number and κ is the integer given by $N_i = \kappa l_i^\psi$.

The procedure described above was used to obtain the solitons solutions of the CAT and AT models associated to any simple Lie algebra [5]. For simplicity we have chosen to discuss here the examples of $SU(6)$ and $Sp(3)$ which possess all the features of the procedure.

4 The Example of $SU(6)$

The L matrix in this case coincides with the extended Cartan matrix since $l_i^\psi = 1$ for any $i = 0, 1, \dots, 5$ and is written as

$$K = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \quad (4.1)$$

and Hirota's equation (3.2) reads

$$\tau_j^2 \triangle (\tau_j) = \beta (\tau_j^2 - \tau_{j+1} \tau_{j-1}) \quad \text{for } j = 0, 1, 2, \dots, r \quad (4.2)$$

where, due to the periodicity of the extended Dynkin diagram, it is understood that $\tau_{j+6} = \tau_j$.

Remember that the $\delta^{(1)}$'s are obtained from the first order term of Hirota's expansion as the eigenvectors of L . The eigenvalues of L are:

$$\lambda_j = 4 \sin^2 \left(\frac{j\pi}{6} \right) \quad \text{for } j = 0, 1, 2, \dots, 5 \quad (4.3)$$

or $\lambda = (0, 1, 3, 4, 3, 1)$ and the corresponding eigenvectors are

$$v_k = 1 \quad \text{for } \lambda = 0$$

$$\begin{aligned}
v_{[1]k} &= \exp\left(\frac{2\pi ik}{6}\right) \quad ; \quad v_{[2]k} = \exp\left(-\frac{2\pi ik}{6}\right) \quad \text{for } \lambda = 1 \\
v_{[1]k} &= \exp\left(\frac{2\pi ik}{3}\right) \quad ; \quad v_{[2]k} = \exp\left(-\frac{2\pi ik}{3}\right) \quad \text{for } \lambda = 3 \\
v_k &= (-1)^k \quad \text{for } \lambda = 4
\end{aligned} \tag{4.4}$$

where $k = 0, 1, 2, \dots, 5$

The function $V^{(n-1)}$, appearing in (3.12), is obtained from (4.2) and for the algebra which is being considered has the form :

$$V_j^{(n-1)} = - \sum_{l=1}^n \left(\left(1 - \lambda (n^2 - 3nl + 2l^2) \right) \delta_j^{(l)} \delta_j^{(n-l)} - \delta_{j+1}^{(n-l)} \delta_{j-1}^{(l)} \right) \tag{4.5}$$

For $\lambda = 0$ one gets a trivial solution where all φ 's are constant. For $\lambda = 4$, $\delta_j^{(1)} = (-1)^j$ and $V_j^{(1)} = 0$. The series truncates at first order, so the τ function is given by:

$$\tau_j = 1 + (-1)^j e^\Gamma \tag{4.6}$$

Substituting (4.6) into (3.1) one gets the explicit expression for the φ 's:

$$\varphi^a = \frac{1}{\bar{q}} \left(-\ln \left(\frac{1 + (-1)^a e^\Gamma}{1 + e^\Gamma} \right) + \vartheta_a \right) \tag{4.7}$$

and the mass is, according to (3.15) $M = \frac{48}{\psi^2} m$.

When $\lambda = 3$ there is a degeneracy and $\delta^{(1)}$ must be a linear combination of its corresponding eigenvectors. The recurrence method furnishes:

$$\tau_j = 1 + \left(y_1 \exp\left(\frac{2\pi ij}{3}\right) + y_2 \exp\left(-\frac{2\pi ij}{3}\right) \right) e^\Gamma + \frac{1}{4} y_1 y_2 e^{2\Gamma} \tag{4.8}$$

where y_1 and y_2 are free parameters and the masses are:

$$M_l^{\lambda=3} = \frac{24\sqrt{3}}{\psi^2} m \quad \text{for } y_1 = 0 \text{ or } y_2 = 0 \tag{4.9}$$

$$M_2^{\lambda=3} = \frac{48\sqrt{3}}{\psi^2} m \quad \text{for } y_1, y_2 \neq 0 \tag{4.10}$$

The last eigenvalue, $\lambda = 1$, is also degenerate. Following the same steps as for the case $\lambda = 3$, $\delta^{(1)}$ must be a linear combination of the corresponding eigenvectors. However, in this case it will appear the second type degeneracy (3.14), since 4 is also an eigenvalue: $4 = 2^2 \cdot 1$. Due to this fact $\delta^{(2)}$ is not uniquely determined from $\delta^{(1)}$. The τ function is now written as:

$$\begin{aligned}
\tau_j &= 1 + \left(y_1 \exp\left(\frac{i\pi j}{3}\right) + y_2 \exp\left(-\frac{i\pi j}{3}\right) \right) e^\Gamma + \left(\frac{3}{4} y_1 y_2 + (-1)^j z \right) e^{2\Gamma} \\
&+ \frac{z}{3} (-1)^j \left(y_1 \exp\left(\frac{i\pi j}{3}\right) + y_2 \exp\left(-\frac{i\pi j}{3}\right) \right) e^{3\Gamma} + (-1)^j \frac{z y_1 y_2}{12} e^{4\Gamma}
\end{aligned} \tag{4.11}$$

and the masses are:

$$M_1^{\lambda=1} = \frac{24}{\psi^2} m \quad \text{for } z = 0 \text{ and } y_1 = 0 \text{ or } y_2 = 0 \quad (4.12)$$

$$M_2^{\lambda=1} = \frac{48}{\psi^2} m \quad \text{for } z = 0 \text{ and } y_1, y_2 \neq 0 \text{ or } z \neq 0 \text{ and } y_1 = y_2 = 0 \quad (4.13)$$

$$M_3^{\lambda=1} = \frac{72}{\psi^2} m \quad \text{for } z \neq 0 \text{ and } y_1 = 0 \text{ or } y_2 = 0 \quad (4.14)$$

$$M_4^{\lambda=1} = \frac{96}{\psi^2} m \quad \text{for } z, y_1, y_2 \neq 0 \quad (4.15)$$

5 The Example of $Sp(3)$

The Cartan matrix is given by

$$K = \begin{pmatrix} 2 & -2 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix} \quad (5.1)$$

$l_i^\psi = 1$ for $i = 0, 1, 2, 3$, and yields the following Hirota's equations

$$\begin{aligned} \tau_0^2 \triangle (\tau_0) &= \beta (\tau_0^2 - \tau_1^2) \\ \tau_1^2 \triangle (\tau_1) &= \beta (\tau_1^2 - \tau_2 \tau_0) \\ \tau_2^2 \triangle (\tau_2) &= \beta (\tau_2^2 - \tau_1 \tau_3) \\ \tau_3^2 \triangle (\tau_3) &= \beta (\tau_3^2 - \tau_2^2) \end{aligned} \quad (5.2)$$

One notices that such equations can be obtained from Hirota's eqs. (4.2) for $SU(6)$ by making the identifications $\tau_5 \equiv \tau_1$ and $\tau_4 \equiv \tau_2$. Therefore, all solutions of (4.2) which are invariant under the interchanges $\tau_5 \leftrightarrow \tau_1$ and $\tau_4 \leftrightarrow \tau_2$ lead to solutions of (5.2). It turns out that such procedure leads to all soliton solutions of $Sp(3)$ (in fact this generalizes to any $SU(2N)$ and $Sp(N)$ [5]).

For instance, the solution (4.11) possesses such symmetry for $y_1 = y_2 \equiv y/2$. Therefore the corresponding solution for $Sp(3)$ is given by

$$\tau_j = 1 + y \cos\left(\frac{\pi j}{3}\right) e^\Gamma + \left(\frac{3}{16} y^2 + (-1)^j z\right) e^{2\Gamma} + \frac{z}{3} y \cos\left(\frac{2\pi j}{3}\right) e^{3\Gamma} + (-1)^j \frac{zy^2}{48} e^{4\Gamma} \quad (5.3)$$

for $j = 0, 1, 2, 3$. The masses are given by

$$M_1^{\lambda=1} = \frac{48}{\psi^2} m \quad \text{for } z = 0 \text{ and } y \neq 0 \text{ or } z \neq 0 \text{ and } y = 0 \quad (5.4)$$

$$M_2^{\lambda=1} = \frac{96}{\psi^2} m \quad \text{for } z, y \neq 0 \quad (5.5)$$

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